Lesson 5. Equations of Lines and Planes in 3D

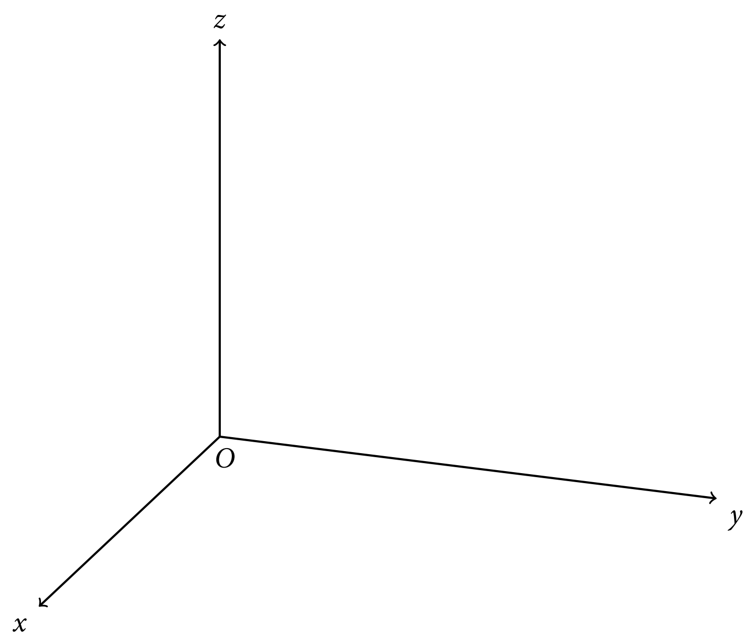
1 In this lesson...

- Different ways of writing equations of lines and planes in 3D

2 Equations of lines in 3D

2.1 Vector equations of lines

- A **line** L is determined by a point $P_0(x_0, y_0, z_0)$ and a direction given by a vector $\vec{v}$



- The **position vector** of a point $P(a_1, a_2, a_3)$ is the vector from the origin $O(0,0,0)$ to the point P
- Let $\vec{r}_0$ be the position vector of P_0: that is, $\vec{r}_0 =$ $\boxed{}$
- The position vector of every point on L can be expressed as the sum of $\vec{r}_0$ and a scalar multiple of $\vec{v}$
- The **vector equation** of line L is $\boxed{}$
  - Each value of the **parameter** t gives a position vector $\vec{r}$ on the line L
  - Positive values of $t \Leftrightarrow$ points on one side of P_0
  - Negative values of $t \Leftrightarrow$ points on the other side of P_0

Example 1.

a. Find a vector equation for the line that passes through the point $(2,4,3)$ and is parallel to the vector $\vec{i}-2\vec{j}+4\vec{k}$.
b. Find two other points on the line.

2.2 Parametric equations of lines

- Suppose $\vec{r(t)} = \langle x(t), y(t), z(t)\rangle$, $\vec{v} = \langle a, b, c\rangle$
- So, we can write the vector equation $\vec{r} = \vec{r}_0 + t\vec{v}$ as

$$\langle x(t), y(t), z(t)\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

- The **parametric equations** of line L are
- The numbers a, b, c are called the **direction numbers** of line L

Example 2. Find a set of parametric equations for the line described in Example 1.

2.3 Symmetric equations of lines

- By solving the parametric equations to eliminate t, we obtain the **symmetric equations** of line L:

Example 3. Find symmetric equations for the line through $(2, -1, 1)$ and perpendicular to both $\vec{u} = \langle 1, 0, 1\rangle$ and $\vec{v} = \langle -1, 1, 0\rangle$.

2.4 Equations of a line in 3D are not unique

- We can use any point on the line as the starting point $P_0 = (x_0, y_0, z_0)$
- We can also use any vector parallel to the line as the direction vector $\vec{v} = \langle a, b, c\rangle$

Example 4. In Example 1, we considered a line that passes through the point $(2, 4, 3)$ and is parallel to the vector $\vec{i} - 2\vec{j} + 4\vec{k}$.

a. Using a different point, find another set of parametric equations for this line.
b. Using a different direction vector, find another set of parametric equations for this line.

2.5 Parallel lines and skew lines

- Two lines are **parallel** if their directions are given by parallel vectors
- Two lines are **skew lines** if they do not intersect and are not parallel
  - i.e., they do not lie on the same plane

Example 5. Here are parametric equations for two lines:

$$L_1 : \quad \left\{ \begin{aligned} x &= 1 + t \\ y &= -2 + 3t \\ z &= 4 - t \end{aligned} \right\} \qquad\qquad L_2 : \quad \left\{ \begin{aligned} x &= 2s \\ y &= 3 + s \\ z &= -3 + 4s \end{aligned} \right\}$$

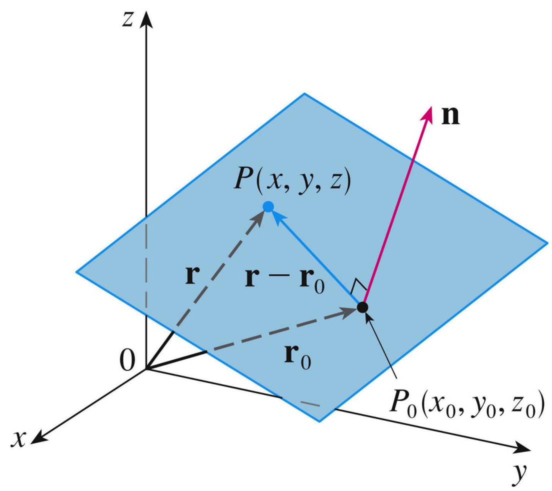
Are they parallel? Are they skew lines?

3 Equations of planes in 3D

3.1 Vector and scalar equations of planes

- A **plane** is determined by
  - a point $P_0(x_0, y_0, z_0)$ on the plane and
  - a **normal vector** $\vec{n}$ orthogonal to the plane
- Let $\vec{r}_0$ be the position vector of P_0; that is, $\vec{r}_0 =$
- Let $\vec{r}$ be the position vector of some point on the plane, say $\vec{r} = \langle x, y, z\rangle$

$\Rightarrow$ $\vec{r} - \vec{r}_0$ is a vector in the plane, and must be orthogonal to the normal vector $\vec{n}$

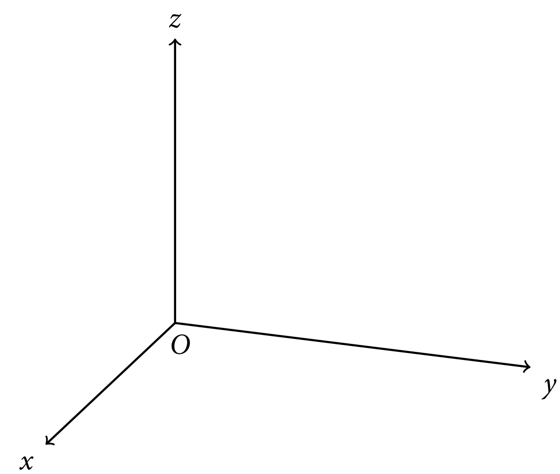


- The **vector equation** of the plane is
- Let $\vec{n} = \langle a, b, c\rangle$
- Expanding the vector equation, we obtain

- The **scalar equation** of the plane is

Example 6.

a. Find an equation of the plane through the point $(-1, 4, 2)$ with normal vector $\vec{n} = \langle 4, 3, 2\rangle$.
b. Find where the plane intercepts the x-, y- and z-axes. Sketch the plane in the first orthant.



Example 7. Find an equation of the plane that passes through the point $(1, 2, 3)$ and is perpendicular to the line $x = 3t$, $y = 1 + t$, $z = 2 - t$.

4 Practice!

Example 8. Find an equation of the plane that passes through the points $P(1, 2, 3)$, $Q(3, 6, -1)$ and $R(5, 0, 2)$.

Example 9. Find the point at which the line

$$x = 1 + 2t, \quad y = 4t, \quad z = 2 - 3t$$

intersects the plane $x + 2y - z + 1 = 0$.

Example 10. Find an equation of the plane that passes through the point $(1, 2, 3)$ and contains the line $x = 3t$, $y = 1 + t$, $z = 2 - t$.